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A new extended Caputo fractional derivative operator associated with Hypergeometric function

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Abstract

In this paper, we obtain a new extended Caputo fractional derivative operator, extension of hypergeometric function and their integral representation. Moreover, we establish some new generating relations involving the extended hypergeometric function & the Apell hypergeometric function by mainly applying power series expansion.

Keywords :- Apell hypergeometric function, Caputo fractional derivative operator, generalized hypergeometric function, , generating relations.

Ams subject classification:-26A33, 33C05

1. Introduction: The Guass hypergeometric function [5] as

$$ {}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \qquad |z| < 1 \tag{1.1}$$

$$(\mathrm{a,b,c} \in \mathbb{C} \; and \; a,b,c \; \neq 0,-1,-2,\ldots)$$

The integral representation of (1.1) is

$$ {}_2F_1(a,b,c,z) = \frac{\Gamma c}{\Gamma b \Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt \tag{1.2}$$

where $\Re(c) > R(b) > 0$ and $|\arg(1-z)| < \pi$.

Confluent hypergeometric function

$$\phi(b;c;z) = \sum_{n=0}^{\infty} \frac{(b)_n}{(c)_n} \frac{z^n}{n!}, \qquad |z| < 1 \; .$$

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The Apell series or bivariat hypergeometric function [5] defined as

$$F_1(a,b,c;d;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m\,(c)_n}{(d)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \tag{1.3}$$

For all

$$(|x| < 1, |y| < 1\mathrm{a,b,c,d} \in \mathbb{C}\ and\ a,b,c\ \neq 0,-1,-2,\dots)$$

The integral representation of (1.3) is

$$F_1(a,b,c;d;x,y) = \frac{\Gamma d}{\Gamma a \Gamma(d-a)} \int_0^1 t^{a-1}\,(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}dt \tag{1.4}$$

Where $\Re(d) > R(a) > 0$ and $|\arg(1-x)| < \pi\ and\ |\arg(1-y)| < \pi$.

The extended hypergeometric and Confluent hypergeometric function is given [6] as follows

$$F_p(a,b;c;z) = \sum_{n=0}^{\infty} \frac{B_p(b+n,c-b)(a)_n}{B(b,c-b)} \frac{z^n}{n!} \tag{1.5}$$

and

$$\phi_p(b;c;z) = \sum_{n=0}^{\infty} \frac{B_p(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \quad , \qquad p \geq 0. \tag{1.6}$$

Integral representation of (1.5) and (1.6) respectively defined as

$$F_p(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-b-1}(1-zt)^{-a} e^{\frac{-p}{t(1-t)}}dt \tag{1.7}$$

Where $p \geq 0$, $\Re(c) > R(b) > 0$ and $|\arg(1-z)| < \pi$

and

$$\phi_p(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-b-1} \exp(zt - \frac{(-p)}{t(1-t)})dt$$

. (1.8)

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The extended Apell's function [8] is given below

$$F_{1,P}(\,a,b,c;d;x,y) = \sum_{\substack{m=0,\\ n=0}}^{\infty} \frac{B_p(a+m+n,d-a)(b)_m\,(c)_n}{B(a,d-a)} \frac{x^m}{m!} \frac{y^n}{n!} \;\;,\;\; p \geq 0$$

(1.9)

Integral representation of (1.9) is

$$F_{1,P}(\,a,b,c;d;x,y) = \frac{1}{B(a,d-a)} \int_0^1 t^{a-1}\,(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c} e^{\frac{-p}{t(1-t)}} dt$$

(1.10)

Where $p \geq 0, \Re(d) > R(a) > 0$ and $|\arg(1-x)| < \pi\, and\, |\arg(1-y)| < \pi$.

We put p=0 in the equation (1.5) & (1.9), reduces to equation (1.1) & (1.3) respectively.

The new modified hypergeometric function and integral representation is

$$F_p^{(\alpha,\beta;\gamma)}(\,a,b;c;z) = \sum_{n=0}^{\infty} \frac{B_p^{(\alpha,\beta;\gamma)}(b+n,c-b)(a)_n}{B(b,c-b)} \frac{z^n}{n!}$$

. (1.11)

Where

$$B_p^{(\alpha,\beta;\gamma)}(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1} \frac{1}{\Gamma\gamma}\, {}_1\psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix}; \frac{-p}{t(1-t)}\right] dt$$

(1.12)

$p, \alpha \geq 0$, $a, b, c \in \mathbb{C}$ and $|z| < 1$.

$$F_p^{(\alpha,\beta;\gamma)}(\,a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-b-1}(1-zt)^{-a} \frac{1}{\Gamma\gamma}\, {}_1\psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix}; \frac{-p}{t(1-t)}\right] dt$$

(1.13)

where $\Re(p) > 0, \Re(c) > R(b) > 0, \min[\Re(\alpha), \Re(\beta), \Re(\gamma)] > 0$.

In eq.(1.11) $\gamma = 1$, reduces in extension of generalized gauss hypergeometric function[8] and $\gamma = 1, \beta = 1$, reduces in extension of gauss hypergeometric function[7].

If we put α=β=γ=1 then eq. (1.11) reduces to the eq. (1.5).

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Generalization of Apell hypergeometric function and integral representation given as below

$$F_{1,p}^{(\alpha,\beta;\gamma)}(a,b,c;d;x,y) = \sum_{\substack{m=0,\\ n=0}}^{\infty} \frac{B_p^{(\alpha,\beta;\gamma)}(a+m+n,d-a)(b)_m\,(c)_n}{B(a,d-a)} \frac{x^m}{m!}\frac{y^n}{n!}$$

$$= \frac{1}{B(a,d-a)} \int_0^1 t^{a-1}\,(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}\frac{1}{\Gamma\gamma}\,{}_1\psi_1\left[\frac{(\gamma,1)}{(\beta,\alpha)};\frac{-p}{t(1-t)}\right]dt$$

(1.14)

where $\Re(p) > 0, \Re(c) > R(b) > 0, \min[\Re(\alpha), \Re(\beta), \Re(\gamma)] > 0, |\arg(1-x)| < \pi$

and $|\arg(1-y)| < \pi$.

Generalization of Confluent hypergeometric function and integral representation given as

$$\phi_p^{(\alpha,\beta;\gamma)}(b;c;z) = \sum_{n=0}^{\infty} \frac{B_p^{(\alpha,\beta;\gamma)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}$$

$$= \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-b-1}\frac{1}{\Gamma\gamma}\,{}_1\psi_1\left[\frac{(\gamma,1)}{(\beta,\alpha)};(zt-\frac{(-p)}{t(1-t)})\right]dt$$

(1.15)

$\Re(p) > 0, \Re(c) > R(b) > 0, \min[\Re(\alpha), \Re(\beta), \Re(\gamma)] > 0.$

2. Extension of hypergeometric functions:

Definition 2.1 The extension of hypergeometric function is defined as

$${}_2F_1,p^{(\alpha,\beta;\gamma)}(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(b-m)_n}.\frac{B_p^{(\alpha,\beta;\gamma)}(b-m+n,c+m-b)}{B(b-m,c-b+m)}.\frac{z^n}{n!}$$

(2.1)

Where$\Re(p) > 0, \min[\Re(\alpha), \Re(\beta), \Re(\gamma)] > 0$, m $- 1 < R(b-c) < m < R(b) > 0$ and $|z| < 1$.

Definition 2.2 The extension of Apell hypergeometric function is defined as

$$F_{1,p}^{(\alpha,\beta;\gamma)}(a,b;c;d;z,z) = \sum_{\substack{n=0\\ k=0}}^{\infty} \frac{(b)_n(c)_k(a)_{n+k}}{(a-m)_{n+k}}.\frac{B_p^{(\alpha,\beta;\gamma)}(a+m+n,d-a)}{B(a-m,d+m-a)}.\frac{z^n}{n!}.\frac{z^k}{k!}$$

(2.2)

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The integral representation of (2.1) & (2.2) are given respectively as

$$ {}_2F_1,p^{(\alpha,\beta;\gamma)}(a,b;c;z) = \frac{1}{B(b-m,c-b+m)} $$

$$ \times \int_0^1 t^{b-m+1}(1-t)^{c-b+m-1}\frac{1}{\Gamma\gamma}\,{}_1\psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};\frac{-p}{t(1-t)}\right]{}_2F_1(a,b,c-m;zt)dt \tag{2.3} $$

and

$$ F_{1,p}^{(\alpha,\beta;\gamma)}(a,b;c;d;z,z) = \frac{1}{B(a-m,c-a+m)} $$

$$ \times \int_0^1 t^{a-m-1}(1-t)^{d-a-m-1}\frac{1}{\Gamma\gamma}\,{}_1\psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};\frac{-p}{t(1-t)}\right]F_1(a,b;c;a-m;zt;zt)dt\,. \tag{2.4} $$

Remark 2.1 If we put α=β=γ=1, then eq. (2.1) to (2.4) reduces to the hypergeometric function ${}_2F_1$ and Apell hypergeometric function F_1, their integral representation [4] respectively.

3. Extension of fractional derivative operator:

Reaserch papers [1], [2], [4], introduced various extension and generalization of fractional calculus. In this section, we define a new extension of extended Caputo fractional derivative operator.

Definition 3.1 The classical Caputo fractional derivative operator which is defined by [6]

$$ D_z^{\mu}[f(z)] = \frac{1}{\Gamma(m-\mu)}\int_0^z (z-t)^{(m-\mu-1)}\frac{d^m}{dt^m}f(t)dt \tag{3.1} $$

where m-1<$\Re(\mu) < m$, m=1,2,3…., .

Definition 3.2 Kiyamaz [4] introduced the extended Caputo fractional derivative operator as

$$ D_z^{\mu}[f(z);p] = \frac{1}{\Gamma(m-\mu)}\int_0^z (z-t)^{(m-\mu-1)}exp(\frac{-p}{t(1-t)})\frac{d^m}{dt^m}f(t)dt \tag{3.2} $$

where m-1<$\Re(\mu) < m$, m=1,2,3…., $\Re(p) > 0$.

Definition 3.3 Recently G.Rahman [7] defined the new extension of extended Caputo fractional derivative operator as

$$ D_z^{\mu}[f(z);p,\alpha] = D_{z,p}^{\mu;\alpha}[f(z)] = \frac{1}{\Gamma(m-\mu)}\int_0^z (z-t)^{(m-\mu-1)}\frac{1}{\Gamma\gamma}\,{}_1\psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};\left(\frac{-pz^2}{t(z-t)}\right)\right]\frac{d^m}{dt^m}f(t)dt \tag{3.3} $$

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For above m-1<$\Re(\mu) < m$, m=1,2,3…., $\Re(p) > 0$.

Definition 3.4 A new extended form of (3.3) is defined as below

$$D_z^{\mu}[f(z);p,\alpha,\beta;\gamma] = D_{z,p}^{\mu;\alpha,\beta;\gamma}[f(z)] = \frac{1}{\Gamma(m-\mu)}\int_0^z (z-t)^{(m-\mu-1)}\frac{1}{\Gamma\gamma}\,{}_1\psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};\left(\frac{-pz^2}{t(z-t)}\right)\right]\frac{d^m}{dt^m}f(t)dt \tag{3.4}$$

m-1<$\Re(\mu) < m$, m=1,2,3…., $\Re(p) > 0$, $\min[\Re(\alpha),\Re(\beta),\Re(\gamma)] > 0$.

Theorem 3.1: The following formula holds true

$$D_z^{\mu}[z^{\eta};p,\alpha,\beta;\gamma] = \frac{\Gamma(\eta+1)B_p^{(\alpha,\beta;\gamma)}(\eta-m+1,m-\mu)}{\Gamma(\eta-\mu+1)B(\eta-m+1,m-\mu)}z^{\eta-\mu} \tag{3.5}$$

Where m-1<$\Re(\mu) < m$, m=1,2,3…., $\Re(p) > 0$, $\min[\Re(\alpha),\Re(\beta),\Re(\gamma)] > 0$, $\Re(\mu) < R(\eta)$, $\Re(\mu) > 0$.

Proof: By using (3.4), we obtain

$$D_{z,p}^{\mu;\alpha,\beta;\gamma}[z^{\eta}] = \frac{1}{\Gamma(m-\mu)}\int_0^z (z-t)^{(m-\mu-1)}\frac{1}{\Gamma\gamma}\,{}_1\psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};\left(\frac{-pz^2}{t(z-t)}\right)\right]\frac{d^m}{dt^m}t^{\eta}dt$$

$$= \frac{1}{\Gamma(m-\mu)}[\eta(\eta-1)\,\ldots.(\eta-m+1)]\int_0^z (z-t)^{(m-\mu-1)}\frac{1}{\Gamma\gamma}\,{}_1\psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};\left(\frac{-pz^2}{t(z-t)}\right)\right]t^{\eta-m}dt \tag{3.6}$$

Substituting t=uz in (3.6), we have obtain

$$D_{z,p}^{\mu;\alpha,\beta;\gamma}[z^{\eta}] = \frac{\Gamma(\eta+1)z^{\eta-\mu}}{\Gamma(\eta-m+1)\Gamma(m-\mu)}\int_0^z u^{\eta-m}(1-u)^{(m-\mu-1)}\frac{1}{\Gamma\gamma}\,{}_1\psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};\frac{-p}{u(1-u)}\right]du$$

By using the definition (1.12), we have

$$= \frac{\Gamma(\eta+1)\Gamma(\eta-\mu+1)}{\Gamma(\eta-\mu+1)\Gamma(\eta-m+1)\Gamma(m-\mu)}B_p^{(\alpha,\beta;\gamma)}(\eta-m+1,m-\mu)z^{\eta-\mu}$$

$$= \frac{\Gamma(\eta+1)B_p^{(\alpha,\beta;\gamma)}(\eta-m+1,m-\mu)}{\Gamma(\eta-\mu+1)B(\eta-m+1,m-\mu)}z^{\eta-\mu}$$

which is the required result.

Theorem 3.2: Let m-1<$\Re(\mu) < m$ and suppose that the function *f(z)* is analytic on the disk $|z| < r$ for some $r \in \mathbb{R}^+$ and with its power series expansion given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then

$$D_z^\mu[f(z);p,\alpha,\beta;\gamma] = \sum_{n=0}^{\infty} a_n D_z^\mu[z^n;p,\alpha,\beta;\gamma] \tag{3.7}$$

Proof: by theorem (3.1)

$$D_z^\mu[f(z);p,\alpha,\beta;\gamma] = \sum_{n=0}^{\infty} \frac{\Gamma(n+1)B_p^{(\alpha,\beta;\gamma)}(n-m+1,m-\mu)a_n}{\Gamma(n-\mu+1)B(n-m+1,m-\mu)} z^{n-\mu}$$

$$D_z^\mu[f(z);p,\alpha,\beta;\gamma] = \sum_{n=0}^{\infty} a_n D_z^\mu[z^n;p,\alpha,\beta;\gamma]$$

Theorem 3.3: Let m-1<$\Re(\mu) < m$ and suppose that the function *f(z)* is analytic on the disk $|z| < r$ for some $r \in \mathbb{R}^+$ and with its power series expansion given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then

$$D_z^\mu\left[z^{\delta-1}f(z);p,\alpha,\beta;\gamma\right] = \frac{\Gamma(\delta)\, z^{\delta-\mu-1}}{\Gamma(\delta-\mu)} \sum_{n=0}^{\infty} \frac{a_n(\delta)_n B_p^{(\alpha,\beta;\gamma)}(\delta-m+n,m-\mu)}{(\delta-m)_n B(\delta-m,m-\mu)} z^n \tag{3.8}$$

Proof: By applying theorem (3.2) & (3.1)

$$D_z^\mu\left[z^{\delta-1}f(z);p,\alpha,\beta;\gamma\right] = \sum_{n=0}^{\infty} a_n D_z^\mu\left[z^{\delta+n-1};p,\alpha,\beta;\gamma\right]$$

$$= \sum_{n=0}^{\infty} \frac{a_n\Gamma(\delta+\mathrm{n})B_p^{(\alpha,\beta;\gamma)}(\delta-m+n,m-\mu)}{\Gamma(\delta+\mathrm{n}-\mu)B(\delta-m+n,m-\mu)} z^{\delta+n-\mu}$$

$$= \sum_{n=0}^{\infty} \frac{a_n(\delta)_n\Gamma(\delta)\, B_p^{(\alpha,\beta;\gamma)}(\delta-m+n,m-\mu)}{(\delta-m)_n\Gamma(\delta-\mathrm{m})\Gamma(\mathrm{m}-\mu)} z^{\delta+n-\mu-1}$$

$$= \sum_{n=0}^{\infty} \frac{a_n(\delta)_n\Gamma(\delta)\,\Gamma(\delta-\mu)\, B_p^{(\alpha,\beta;\gamma)}(\delta-m+n,m-\mu)}{(\delta-m)_n\Gamma(\delta-\mu)\,\Gamma(\delta-\mathrm{m})\Gamma(\mathrm{m}-\mu)} z^{\delta+n-\mu-1}$$

$$= \frac{\Gamma(\delta)z^{\delta-\mu-1}}{\Gamma(\delta-\mu)} \sum_{n=0}^{\infty} \frac{a_n(\delta)_n B_p^{(\alpha,\beta;\gamma)}(\delta-m+n,m-\mu)}{(\delta-m)_n B(\delta-m,m-\mu)} z^n$$

which is the required result.

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Theorem 3.4: The following formula holds true

$$D_z^{\delta-\mu}\left[z^{\delta-1}(1-z)^{-\lambda};p,\alpha,\beta;\gamma\right] = \frac{\Gamma(\delta)\, z^{\mu-1}}{\Gamma(\mu)}\sum_{n=0}^{\infty}\frac{(\lambda)_n(\delta)_n B_p^{(\alpha,\beta;\gamma)}(\delta-m+n,m-\delta+\mu)}{(\delta-m)_n B(\delta-m,\mu-\delta+m)}\frac{z^n}{n!}$$

$$= \frac{\Gamma(\delta)\, z^{\mu-1}}{\Gamma(\mu)}\, {}_2F_1, p^{(\alpha,\beta;\gamma)}(\lambda,\delta;\mu;z) \qquad (3.9)$$

Where m-1<$\Re(\delta-\mu) < m < R(\delta) > 0$, m=1, 2, 3…., $\Re(p) > 0$, min$[\Re(\alpha),\Re(\beta),\Re(\gamma)] > 0$

Proof: By using power series $(1-z)^{-\lambda}$ and applying theorem (3.1), we have obtain

$$D_z^{\delta-\mu}\left[z^{\delta-1}(1-z)^{-\lambda};p,\alpha,\beta;\gamma\right] = D_z^{\delta-\mu}\left[z^{\delta-1}\sum_{n=0}^{\infty}(\lambda)_n\frac{z^n}{n!};p,\alpha,\beta;\gamma\right]$$

$$= \left[\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\, D_z^{\delta-\mu}\{z^{\delta+n-1};p,\alpha,\beta;\gamma\}\right]$$

$$= \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\frac{\Gamma(\delta+n)B_p^{(\alpha,\beta;\gamma)}(\delta-m+n,m-\delta+\mu)}{\Gamma(\delta-m+n)\Gamma(m-\delta+\mu)}z^{\mu+n-1}$$

$$= \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\frac{(\delta)_n\Gamma(\delta)\; B_p^{(\alpha,\beta;\gamma)}(\delta-m+n,m-\delta+\mu)}{(\delta-m)_n\Gamma(\delta-m)\;\Gamma(m-\delta+\mu)}z^{\mu+n-1}$$

$$= \frac{\Gamma(\delta)\, z^{\mu-1}}{\Gamma(\mu)}\sum_{n=0}^{\infty}\frac{(\lambda)_n(\delta)_n B_p^{(\alpha,\beta;\gamma)}(\delta-m+n,m-\delta+\mu)}{(\delta-m)_n B(\delta-m,\mu-\delta+m)}\frac{z^n}{n!}$$

$$= \frac{\Gamma(\delta)\, z^{\mu-1}}{\Gamma(\mu)}\, {}_2F_1, p^{(\alpha,\beta;\gamma)}(\lambda,\delta;\mu;z)$$

which is the required result.

Theorem 3.5: The following formula holds true

$$D_z^{\delta-\mu}\left[z^{\delta-1}(1-az)^{-\lambda}(1-bz)^{-\eta};p,\alpha,\beta;\gamma\right]$$
$$= \frac{\Gamma(\delta)\, z^{\mu-1}}{\Gamma(\mu)}\sum_{\substack{n=0\\k=o}}^{\infty}\frac{(\lambda)_n(\eta)_k(\delta)_{n+k}B_p^{(\alpha,\beta;\gamma)}(\delta+m+n+k,\mu-\delta)}{(\delta-m)_{n+k}B(\delta-m,m+\mu-\delta)}\frac{(az)^n}{n!}\frac{(bz)^k}{k!}$$

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$$= \frac{\Gamma(\delta)\, z^{\mu-1}}{\Gamma(\mu)} F_{1,p}^{(\alpha,\beta;\gamma)}(\,\delta; \lambda, \mu; az; bz) \tag{3.10}$$

Where m-1<$\Re(\delta-\mu) < m < R(\delta) > 0$, m=1, 2, 3…., $\Re(p) > 0$, min$[\Re(\alpha), \Re(\beta), \Re(\gamma)] > 0$

Proof: Power series expansion

$$(1-az)^{-\lambda}(1-bz)^{-\eta} = \sum_{\substack{n=0\\k=0}}^{\infty} (\lambda)_n (\eta)_k \frac{(az)^n}{n!} \frac{(bz)^k}{k!}$$

By above expansion, we have obtain

$$D_z^{\delta-\mu}\left[z^{\delta-1}(1-az)^{-\lambda}(1-bz)^{-\eta}; p, \alpha, \beta; \gamma\right]$$

$$= \sum_{\substack{n=0\\k=0}}^{\infty} (\lambda)_n (\eta)_k \frac{(a)^n}{n!} \frac{(b)^k}{k!} D_z^{\delta-\mu}\left[z^{\delta+n+k-1}; p, \alpha, \beta; \gamma\right]$$

using theorem (3.1)

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} (\lambda)_n (\eta)_k \frac{(a)^n}{n!} \frac{(b)^k}{k!} \frac{\Gamma(\delta+n+k) B_p^{(\alpha,\beta;\gamma)}(\delta+m+n+k, \mu-\delta)}{\Gamma(\delta-m+n+k)\Gamma(m+\mu-\delta)} z^{\mu+n+k-1}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} (\lambda)_n (\eta)_k \frac{(a)^n}{n!} \frac{(b)^k}{k!} \frac{\Gamma(\delta)(\delta)_{n+k}\Gamma(\mu) B_p^{(\alpha,\beta;\gamma)}(\delta+m+n+k, \mu-\delta)}{\Gamma(\mu)(\delta-m)_{n+k}\Gamma(\delta-m)\Gamma(m+\mu-\delta)} z^{\mu+n+k-1}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} (\lambda)_n (\eta)_k \frac{(a)^n}{n!} \frac{(b)^k}{k!} \frac{\Gamma(\delta)(\delta)_{n+k}\Gamma(\mu) B_p^{(\alpha,\beta;\gamma)}(\delta+m+n+k, \mu-\delta)}{\Gamma(\mu)(\delta-m)_{n+k} B(\delta-m, m+\mu-\delta)} z^{\mu+n+k-1}$$

$$= \frac{\Gamma(\delta)\, z^{\mu-1}}{\Gamma(\mu)} \sum_{\substack{n=0\\k=0}}^{\infty} \frac{(\lambda)_n (\eta)_k (\delta)_{n+k} B_p^{(\alpha,\beta;\gamma)}(\delta+m+n+k, \mu-\delta)}{(\delta-m)_{n+k} B(\delta-m, m+\mu-\delta)} \frac{(az)^n}{n!} \frac{(bz)^k}{k!}$$

$$= \frac{\Gamma(\delta)\, z^{\mu-1}}{\Gamma(\mu)} F_{1,p}^{(\alpha,\beta;\gamma)}(\,\delta; \lambda, \mu; az; bz).$$

4. Generating relations:

In this section, we applying theorem (3.4), (3.5) and obtain generating relations for the new extension of hypergeometric function ${}_2F_1, p^{(\alpha,\beta;\gamma)}$ and Apell hypergeometric function $F_{1,p}^{(\alpha,\beta;\gamma)}$.

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Theorem 4.1: Assume that m-1<$\Re(\delta-\mu) < m < R(\delta)$ and $|z| < \min\{1, |1-t|\}$ then

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2F_1, p^{(\alpha,\beta;\gamma)}(\lambda+n, \delta; \mu; z)t^n = (1-t)^{-\lambda} {}_2F_1, p^{(\alpha,\beta;\gamma)}\left(\lambda, \delta; \mu; \frac{z}{1-t}\right) \tag{4.1}$$

Proof: We consider the following series

$$[(1-z)-t]^{-\lambda} = (1-t)^{-\lambda}(1-\frac{z}{1-t})^{-\lambda} = (1-z)^{-\lambda}(1-\frac{t}{1-z})^{-\lambda}$$

Thus, the power series expansion, yields

$$= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}(1-z)^{-\lambda}(\frac{t}{1-z})^n = (1-t)^{-\lambda}(1-\frac{z}{1-t})^{-\lambda}$$

Multiplying both sides by $z^{\delta-1}$ and then applying the operator $D_{z,p}^{\delta-\mu;\alpha,\beta;\gamma}$ on both sides, we have

$$D_{z,p}^{\delta-\mu;\alpha,\beta;\gamma}\left[\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}(1-z)^{-\lambda}(\frac{t}{1-z})^n z^{\delta-1}\right] = D_{z,p}^{\delta-\mu;\alpha,\beta;\gamma}\left[z^{\delta-1}(1-t)^{-\lambda}(1-\frac{z}{1-t})^{-\lambda}\right]$$

Interchanging the order of summation and the operator$D_{z,p}^{\delta-\mu;\alpha,\beta;\gamma}$, we have obtain following result

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} D_{z,p}^{\delta-\mu;\alpha,\beta;\gamma}[(1-z)^{-\lambda-n} z^{\delta-1}](t)^n = (1-t)^{-\lambda} D_{z,p}^{\delta-\mu;\alpha,\beta;\gamma}\left[z^{\delta-1}(1-\frac{z}{1-t})^{-\lambda}\right]$$

By using theorem (3.4), we get the required result

$$= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} 2^{F_{1,p}^{(\alpha,\beta;\gamma)}}(\lambda+n, \delta; \mu; z)t^n = (1-t)^{-\lambda} {}_2F_1, p^{(\alpha,\beta;\gamma)}\left(\lambda, \delta; \mu; \frac{z}{1-t}\right)$$

Theorem 4.2: The following generating relation holds true

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} 2^{F_{1,p}^{(\alpha,\beta;\gamma)}}(\nu-n, \delta; \mu; z)t^n = (1-t)^{-\lambda} F_{1,p}^{(\alpha,\beta;\gamma)}\left(\nu, \lambda, \delta; \mu; \frac{-zt}{1-t}\right) \tag{4.2}$$

where $|t| < \frac{1}{1+|t|}$, $\mathrm{m}-1 < R(\delta-\mu) < m < R(\delta)$.

Proof: By considering the following series identity

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$$[(1-z)t]^{-\lambda} = (1-t)^{-\lambda}(1+\frac{zt}{1-t})^{-\lambda}$$

Power series expansion of above identity

$$= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}(1-z)^n(t)^n = (1-t)^{-\lambda}(1+\frac{zt}{1-t})^{-\lambda}$$

Multiplying both sides by $z^{\delta-1}(1-z)^{-\nu}$ and then applying the operator $D_{z,p}^{\delta-\mu;\alpha,\beta;\gamma}$ on both sides, we have

$$D_{z,p}^{\delta-\mu;\alpha,\beta;\gamma}\left[\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}(1-z)^n(t)^n z^{\delta-1}(1-z)^{-\nu}\right]$$

$$= D_{z,p}^{\delta-\mu;\alpha,\beta;\gamma}\left[z^{\delta-1}(1-z)^{-\nu}(1-t)^{-\lambda}(1+\frac{zt}{1-t})^{-\lambda}\right]$$

Interchanging the order of summation and the operator$D_{z,p}^{\delta-\mu;\alpha,\beta;\gamma}$, then

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} D_{z,p}^{\delta-\mu;\alpha,\beta;\gamma}\left[(1-z)^{-\nu+n} z^{\delta-1}\right](t)^n = (1-t)^{-\lambda} D_{z,p}^{\delta-\mu;\alpha,\beta;\gamma}\left[z^{\delta-1}(1-z)^{-\nu}(1+\frac{zt}{1-t})^{-\lambda}\right]$$

Where $\Re(\delta) > R(\mu) > 0$ and$|zt| < |1-t|$, thus by using theorem (3.5), we obtain the required result

$$= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} 2^{F_{1,p}^{(\alpha,\beta;\gamma)}}(\nu-n,\delta;\mu;z)t^n = (1-t)^{-\lambda} F_{1,p}^{(\alpha,\beta;\gamma)}\left(\nu,\lambda,\delta;\mu;\frac{-zt}{1-t}\right)$$

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